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Characterization of Best Chebyshev Approximations with Prescribed Norm

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1. INTRODUCTION

In recent years there has been considerable interest in studying problems of best Chebyshev approximation in situations where the approximant is not to be freely chosen from a linear subspace but rather from some subset characterized by prescribed constraints. The case of convex constraints has been particularly well studied. (See, for example, the recent review by Lewis [1].) If the norm of the approximant is prescribed, the approximating subset is (without loss of generality) the unit sphere in the subspace. This subset is not convex; however, a straightforward application of the convex theory can be used in those cases where the best unconstrained approximation from the subspace lies outside the unit sphere. (This is because, on an intuitive level, the unit sphere "looks like" the convex unit ball when viewed from outside.) If the best unconstrained approximation lies inside the sphere, new techniques are required. Most of this paper deals with characterization theorems for this interesting case. It should be noted that if the approximating subspace is finite dimensional, existence of best approximations from the unit sphere is guaranteed. Furthermore, examples of nonuniqueness are readily generated.

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2. Problem Definition and Preliminaries

We will be dealing with the space $C[a, b]$ of continuous real-valued functions on a closed interval $[a, b]$. The norm will be the uniform, or Chebyshev, norm; i.e.,

$$\| f \| = \max_{x \in [a,b]} | f(x) |.$$

Let L be a given n-dimensional linear subspace in $C[a, b]$ and let S denote the unit sphere of L; i.e.,

$$S = \{ f \in L : \| f \| = 1 \}.$$

Then, given an element f_0 of $C[a, b]$, we set ourselves the problem of characterizing a best approximation h from S to f; i.e., an element h of S such that

$$\| f_0 - h \| \leqslant \| f_0 - q \|$$

for all q in S.

As indicated in the introduction, the best approximation to f_0 from the subspace L plays a key role in the analysis. Henceforth, we will assume that L is a Haar space (so that that best approximation is unique) and we will denote that approximation by g. To avoid trivial cases we will assume that $f_0 \notin L$.

The following obvious lemma will be found useful.

LEMMA 1. *If f, h_1, h_2 satisfy $\| h_1 - f \| > \| h_2 - f \|$, then for all $t \in [0, 1)$, $\| h_1 - f \| > \| th_1 + (1 - t) h_2 - f \|$.*

As a simple consequence of this lemma, one obtains the precise statement of the qualitative remark in the introduction regarding the $\| g \| > 1$ case.

THEOREM 1. *If the best approximation g to f_0 from L satisfies $\| g \| \geqslant 1$, then the best approximations to f_0 from S coincide with the best approximations to f from $\bar{B}$, the closed unit ball of L.*

Thus, we really are dealing with approximation from a convex set in this case. Conceptually, it is most convenient to consider the problem as a special case of approximation by functions having restricted ranges [3]; namely, restricted to between $u(x) = 1$ and $l(x) = -1$. The following definitions are needed for the understanding of the characterization theorem. We shall also find these concepts crucial for the study of the $\| g \| < 1$ case in the next section.

DEFINITION 1. By the collection of critical points of $h \in L$ (written CRITICAL(h)) is meant the collection of extreme points of $h - f_0$, namely,

$$\text{CRITICAL}(h) = \{x \in [a, b]: |h(x) - f_0(x)| = \| h - f_0 \|\}.$$

DEFINITION 2. By the collection of constraint points of $h \in L$ (written CONSTRAINT(h)) is meant the collection of extreme points of h, namely,

$$\text{CONSTRAINT}(h) = \{x \in [a, b]: |h(x)| = \| h \|\}.$$

DEFINITION 3. A bounded function h has m alternations on $[a, b]$ if and only if there is a strictly increasing sequence $\{x_j\}_{j=1}^{m+1}$ of points in $[a, b]$ satisfying both

(1) $|h(x_1)| = \| h \|$, and

(2) $h(x_{j+1}) = -h(x_j)$, $j = 1,..., m$.

Then the following theorem, a corollary of a theorem given by Taylor [3, p. 243], characterizes a best approximation in the case $\| g \| > 1$.

THEOREM 2. *Suppose that* $\| g \| > 1$. *Then* $h \in S$ *is a best Chebyshev approximation to* f_0 *if and only if the function*

$$\begin{aligned} s(x) &= -\text{sgn}(h(x) - f_0(x)), \qquad x \in \text{CRITICAL}(h), \\ &= -\text{sgn}(h(x)), \qquad x \in \text{CONSTRAINT}(h), \\ &= 0, \qquad \textit{otherwise}, \end{aligned}$$

is either not well defined or has more than $n - 1$ *alternations.*

The basic idea behind this theorem may be stated in terms of "correction functions:"

DEFINITION 4. The function $f \in L$ is a correction function for $h \in L$ with respect to f_0 if and only if $\| h + \epsilon f - f_0 \| < \| h - f_0 \|$ for some $\epsilon > 0$.

A standard argument leads to the following lemma, which we will find useful in the next section.

LEMMA 2. *If* h *and* c *are elements of* L, *then* c *is a correction function for* h *with respect to* f_0 *if and only if*

$$\text{sgn}(c(x)) = -\text{sgn}(h(x) - f_0(x)), \qquad \textit{for all} \quad x \in \text{CRITICAL}(h),$$

where

$$\begin{aligned} \operatorname{sgn}(t) &= 1, && \text{if } t > 0, \\ &= 0, && \text{if } t = 0, \\ &= -1, && \text{if } t < 0. \end{aligned}$$

In constructing correction functions, we will not only make use of the unique interpolation property of a Haar space, but also of the following lemma [2, p. 57].

LEMMA 3. *Let a simple zero of a function f be a zero at which f changes sign, and let a double zero be one at which it does not. Then no function in an n-dimensional Haar space has more than $n - 1$ zeros (in $[a, b]$), counting interior double zeros as two zeros.*

In case $\| g \| > 1$, an approximation h_1 to f_0 from S can be improved if and only if there exists a correction function f whose sign at each $x \in \text{CONSTRAINT}(h_1)$ is opposite from that of h_1 (and whose sign at each $x \in \text{CRITICAL}(h_1)$ is necessarily opposite from that of $h_1 - f_0$ by the definition of correction function). For $\epsilon > 0$ sufficiently small, $h_1 + \epsilon f$ lies in B (the open unit ball of L) and is a better approximation to f_0 than is h_1 (Lemma 1). Thus, $h_2 \in \text{SEGMENT}(g, h_1 + \epsilon f) \cap S$ is also a better approximation to f_0 than is h_1. (The notation $\text{SEGMENT}(g, f)$ denotes the line segment joining g to f.)

In case $\| g \| < 1$, we may hope to obtain results by a similar maneuver. That is, we attempt to find a correction function f whose sign at *some* $x \in \text{CONSTRAINT}(h_1)$ *agrees* with that of h_1 (and whose sign at each $x \in \text{CRITICAL}(h_1)$ is necessarily still opposite from that of $h_1 - f_0$ by the definition of correction function). Then for $\epsilon > 0$ sufficiently small, $h_1 + \epsilon f$ lies *outside* of $\bar{B}$ and is a better approximation to f_0 than is h_1. Thus, $h_2 \in \text{SEGMENT}(h_1 + \epsilon f, g) \cap S$ is also a better approximation to f_0 than is h_1.

It will be convenient to have some designation for the set of points x where the norm of h_1 may be so increased. Hence, we introduce the following definition.

DEFINITION 5. The subset of $[a, b]$ denoted by $\text{INCR}(h, f_0)$ and called the points of $[a, b]$ increasable in h with respect to f_0 is defined by: $x \in \text{INCR}(h, f_0)$ if and only if there exists a correction function f (of h with respect to f_0) satisfying

$$f(x)/h(x) > 0.$$

It turns out that this approach leads to only partial success, in that we are only able to characterize *locally* best approximations. Hence, we need one final definition.

DEFINITION 6. If U is a subset of L and $h \in U$, then h is a locally best approximation to f_0 from U if and only if there exists a neighborhood N of h in L (as a topological space with the norm induced topology) such that h is a best approximation to f_0 from $U \cap N$.

3. CHARACTERIZATION WHEN $\| g \| < 1$

We are assuming throughout this section that $\| g \| < 1$. First we prove a necessary condition for a locally best approximation.

THEOREM 3. *If h is a locally best approximation to f_0 from S, then*

$$\text{CONSTRAINT}(h) \cap \text{INCR}(h, f_0) = \varnothing.$$

Proof. We proceed by proving the contrapositive. Suppose there exists some $z \in \text{CONSTRAINT}(h) \cap \text{INCR}(h, f_0)$; then there is some correction function f of h with respect to f_0 satisfying $f(z)/h(z) > 0$ and for all $\epsilon > 0$ sufficiently small, say $\epsilon < \epsilon_0$, $h + \epsilon f$ is a better approximation to f_0 than is h. But we see that $\| h + \epsilon f \| \geqslant | h(z) + \epsilon f(z) | = 1 + \epsilon | f(z) | > 1$. Hence, $h_\epsilon \in S \cap \text{SEGMENT}(h + \epsilon f, g)$ is a better approximation to f_0 than is h (see Lemma 1).

Now, h_ϵ is a convex combination of g and $h + \epsilon f$, say $h_\epsilon = t_\epsilon(h + \epsilon f) + (1 - t_\epsilon) g$, for some $t_\epsilon \in (0, 1)$. Letting $\epsilon_n = \epsilon_0/n$ for each positive integer n, we have that $\{t_{\epsilon_n}\}$ is an infinite sequence in $[0, 1]$ and so contains a convergent subsequence. Relabeling if necessary, we have $\{t_{\epsilon_n}\} \to t \in [0, 1]$ as $n \to \infty$, so $h_{\epsilon_n} \to th + (1 - t) g$. But $1 = \| h_{\epsilon_n} \| \to \| th + (1 - t) g \|$; then Lemma 1 dictates that $t = 1$ lest $\| th + (1 - t) g \|$ be less than 1. Hence, $h_{\epsilon_n} \to h$ as $n \to \infty$, and we see that h is not a locally best approximation to f_0 from S.

Unfortunately, the converse of this theorem is not true. To understand the situation, we first prove the following lemma, which provides some insight into the set $\text{INCR}(h, f_0)$.

LEMMA 4. *In the relative topology of $[a, b]$, $\text{INCR}(h, f_0)$ is an open set whose boundary points x are elements of $\text{CRITICAL}(h) \cup h^{-1}(\{0\})$ and satisfy $h(x)(h(x) - f_0(x)) > 0$ when $h(x) \neq 0$.*

Proof. That $\text{INCR}(h, f_0)$ is open follows simply from the continuity of the functions involved, since if $z \in \text{INCR}(h, f_0)$, then any correction function c increasing h at $x = z$ also increases h in a neighborhood of z (for, by continuity, both c and h must have the same nonzero sign in some neighborhood of z).

Thus, z is a boundary point of $\text{INCR}(h, f_0)$ if and only if $z \notin \text{INCR}(h, f_0)$ but z is the limit of an infinite sequence $\{x_i\}$ in $\text{INCR}(h, f_0)$; therefore, we shall assume the existence of such a sequence and such a boundary point z.

Since $x_j \neq z$ for all j, we shall further assume that $\{x_j\}$ is a strictly increasing sequence, $x_1 < x_2 < \cdots < x_j < \cdots$ with no loss in generality. (The proof for a strictly decreasing sequence is completely analogous.) Let c_j denote a correction function increasing h at $x = x_j$. We shall show that $z \in \text{CRITICAL}(h) \cup h^{-1}(\{0\})$ by contradiction. The idea is that were z not a critical point, then there would be no critical point in some neighborhood of z (relative to $[a, b]$). By choosing x_j in such an interval neighborhood, we could then alter c_j by shifting a simple zero from (x_j, z) to the right of z (or by simply eliminating a simple zero in (x_j, z) if $z = b$), thus obtaining a correction function increasing h with respect to f_0 at z.

Suppose that $z \notin \text{CRITICAL}(h) \cup h^{-1}(\{0\})$. Using the continuity of h and the fact that CRITICAL(h) is closed, we see that there is an $\epsilon > 0$ such that h does not change sign in $(z - \epsilon, z + \epsilon) \cap [a, b]$, and CRITICAL($h$) is disjoint from $(z - \epsilon, z + \epsilon) \cap [a, b]$.

For j sufficiently large, say $j \geqslant J > 0$, $[x_J, z) \subset [z - \epsilon, z]$.

If $c_J(z) = 0$, then we can add to c_J a sufficiently small positive multiple of any function c satisfying $c(z) = h(z)$ so that $c_J + \delta c$ remains a correction function, but now increases h at z. Thus, $z \in \text{INCR}(h, f_0)$, a contradiction.

If $c_J(z)\, h(z) > 0$, then $z \in \text{INCR}(h, f_0)$, a contradiction.

If $c_J(z)\, h(z) < 0$ (the only remaining case since $h(z) \neq 0$), then c_J must have a simple zero z_0 in (x_J, z) (because h has the same sign at x_J as it has at z, yet c_J has opposite signs at x_J and z). Denote the remaining *simple* zeros of c_J by $z_2, z_3, \ldots, z_m$. Using $[\cdot]$ to denote the greatest integer function, set $p = 2[(n - 1 - m)/2]$ and select $p + 2$ distinct points in (x_J, z) that are also distinct from $z_0, z_2, \ldots, z_m$ and label them $z_{m+1}, \ldots, z_{m+p+2}$. Let $q = m + p$; then, either $q = n - 1$, or $q = n - 2$. We are now in a position to construct a correction function c increasing h with respect to f_0 at z, and hence, yielding the desired contradiction. We proceed by cases.

Case I. $z \neq b$, $q = n - 1$. Select $z_1 \in (z, z + \epsilon) \cap (a, b)$ distinct from $z_2, \ldots, z_{n-1}$ and let $c \in L$ be the unique function interpolating zero at $z_1, \ldots, z_{n-1}$ and interpolating $c_J(x_J)$ at x_J. Then by Lemma 3, $z_1, \ldots, z_{n-1}$ are the only zeros of c and they are all simple zeros. Thus, except at double zeros of c_J, c agrees in sign with c_J outside of $(z - \epsilon, z + \epsilon)$ while increasing h at z. By Lemma 2, therefore, c is a correction function increasing h at z.

Case II. $z \neq b$, $q = n - 2$. Select $z_1 \in (z, z + \epsilon) \cap (a, b)$ distinct from $z_2, \ldots, z_{n-2}$, and let $c_0 \in L$ be the function interpolating zero at $z_1, \ldots, z_{n-2}$, interpolating 1 at a and interpolating $(-1)^{n-2}$ at b. Then $c(x) = c_J(x_J)\, c_0(x_J)\, c_0(x)$ is a correction function increasing h at z (because $\text{sgn}(c(x_J)) = \text{sgn}(c_J(x_J))$).

Case III. $z = b$, $q = n - 1$. Let $c \in L$ be the function interpolating zero at $z_2, \ldots, z_{n-1}$, interpolating $c_j(x_j)$ at b, and interpolating $(-1)^{n-2} c_j(x_j)$ at a.

Case IV. $z = b$, $q = n - 2$. Let $c \in L$ be the unique function interpolating zero at $z_2, \ldots, z_n$ and interpolating $c_j(x_j)$ at b.

Since these four cases are exhaustive, we conclude that if $z \notin$ CRITICAL$(h) \cap h^{-1}(\{0\})$, then we obtain the contradiction $z \in$ INCR(h, f_0).

Finally, z satisfies $h(z)(h(z) - f_0(z)) > 0$ when $h(z) \neq 0$ because it satisfies $h(z)(h(z) - f_0(z)) \geqslant 0$ (a result of $z \notin$ INCR(h, f_0)) while $z \in$ CRITICAL(h).

We now prove a theorem that elucidates the chief difficulty in obtaining a converse to Theorem 3.

THEOREM 4. *If $h \in S$ satisfies* CONSTRAINT$(h) \cap$ INCR$(h, f_0) = \varnothing$, *but is not a locally best approximation to f_0 from S, then some $z \in$* CONSTRAINT$(h) \cap$ CRITICAL(h) *is the limit of a sequence $\{x_j\}$ in* INCR(h, f_0) *satisfying*

$$h(z) f_0(z) < h(z) f_0(x_j). \tag{1}$$

Proof. Suppose that $h \in S$, that CONSTRAINT$(h) \cap$ INCR$(h, f_0) = \varnothing$, and that h is not a locally best approximation to f_0 from S.

Part I. Existence of $\{x_j\}$ and z. Since h is not a locally best approximation, there is a sequence $\{h_j\}$ in S converging (uniformly) to h and satisfying $\| h_j - f_0 \| < \| h - f_0 \|$, for all j. For each j, choose $x_j \in$ CONSTRAINT(h_j); then $\{x_j\}$ is an infinite set in a compact interval and so has an accumulation point z. By shifting to a subsequence, we may assume that $\{x_j\}$ converges to z. Now $| h_j(x_j)| = \| h_j \| = \| h \|$; so by the uniformity of the convergence of $\{h_j\}$, $| h(z)| = \| h \|$. Therefore, $z \in$ CONSTRAINT(h).

Part II. $\{x_j\}$ is in INCR(h, f_0). First, $| h(z)| = \| h \| = 1$, so there is some neighborhood N of z such that $h(x)/h(z) > 1/2$ for all $x \in N$. Second, $\{h_j\}$ converges to h uniformly, so there is some integer $M_1 > 0$ for which $h_j(x)/h(z) > 0$ whenever $x \in N$ and $j > M_1$. Third, $\{x_j\}$ converges to z, hence, there is some integer $M_2 \geqslant M_1$ satisfying $x_j \in N$ whenever $j > M_2$.

We can attain these three results simultaneously if we shift to a subsequence of $\{x_j\}$ (and to a corresponding subsequence of $\{h_j\}$) by eliminating the first M_2 elements of the sequence. Doing so, we obtain that for all j,

$$h_j(x_j)/h(z) > 0 \qquad \text{and} \qquad h(x_j)/h(z) > 1/2.$$

Finally, we obtain for all j the sign condition:

$$0 < [h_j(x_j)/h(z)][h(z)/h(x_j)] = h_j(x_j)/h(x_j).$$

Now $h_j - h$ is certainly a correction function because h_j is a better approximation to f_0 than is h. Thus, if $h_j(x_j) - h(x_j) \neq 0$, then $x_j \in \text{INCR}(h, f_0)$ (because $|h_j(x_j)| = 1 \geqslant |h(x_j)|$).

But even if $h_j(x_j) - h(x_j) = 0$, we may take f to be any function of unit norm in L that satisfies $f(x_j)/h_j(x_j) > 0$ and obtain the correction function $c = h_j - h + \frac{1}{2}(\|h - f_0\| - \|h_j - f_0\|)f$. This c has the same sign at x_j as does h, so that $x_j \in \text{INCR}(h, f_0)$.

Part III. $z \in \text{CRITICAL}(h) \cap \text{CONSTRAINT}(h)$. In Part I we obtained $z \in \text{CONSTRAINT}(h)$; we need only show $z \in \text{CRITICAL}(h)$. Now $\text{CONSTRAINT}(h) \cap \text{INCR}(h, f_0) = \varnothing$, so we conclude that $z \notin \text{INCR}(h, f_0)$. But $\{x_j\} \subset \text{INCR}(h, f_0)$ and $\{x_j\} \to z$, so z must be a boundary point of $\text{INCR}(h, f_0)$. Lemma 4 then yields $z \in \text{CRITICAL}(h)$.

Part IV. $h(z) f_0(z) < h(z) f_0(x_j)$. Since $z \in \text{CRITICAL}(h)$ and $h_j - h$ is a correction function for h, we have

$$\operatorname{sgn}(h_j(z) - h(z)) = -\operatorname{sgn}(h(z) - f_0(z)) \neq 0.$$

But $z \in \text{CONSTRAINT}(h)$ and $\|h\| = \|h_j\| \geqslant |h(z) + (h_j(z) - h(z))|$, so

$$\operatorname{sgn}(h_j(z) - h(z)) = -\operatorname{sgn}(h(z)) = -h(z).$$

Thus, $h(z) = \operatorname{sgn}(h(z) - f_0(z))$. Furthermore, for sufficiently large j (say, $j > M_3$)

$$\operatorname{sgn}(h(z) - f_0(z)) = \operatorname{sgn}(h_j(x_j) - f_0(x_j)).$$

By eliminating the first M_3 terms of the sequence $\{h_j\}$ and of the corresponding sequence $\{x_j\}$ and relabeling, we obtain sequences $\{h_j\}$ and $\{x_j\}$ satisfying $h(z) = \operatorname{sgn}(h_j(x_j) - f_0(x_j))$, for all j. Then because h_j is a better approximation to f_0 than is h,

$$\begin{aligned} 0 &< \|h - f_0\| - |h_j(x_j) - f_0(x_j)| \\ &= |h(z) - f_0(z)| - |h_j(x_j) - f_0(x_j)| \\ &= h(z)[h(z) - f_0(z)] - h(z)[h_j(x_j) - f_0(x_j)] \\ &= h(z)[h(z) - h_j(x_j)] - h(z) f_0(z) + h(z) f_0(x_j) \\ &= 0 - h(z) f_0(z) + h(z) f_0(x_j), \end{aligned}$$

which is equivalent to $h(z) f_0(z) < h(z) f_0(x_j)$. This completes the proof of the theorem.

It should be noted that when $h(z)[h(z) - f_0(z)] > 0$ and x_j is sufficiently close to z (so that both h and $h - f_0$ have the same sign at x_j as at z), then (1) is equivalent to saying that the distance from $h(x_j)$ to $\|h\|$ (namely, $\|h\| - |h(x_j)|$) is less than the distance from $h(x_j) - f_0(x_j)$ to $\|h - f_0\|$

(namely, $\| h - f_0 \| - | h(x_j) - f_0(x_j)|$). That is, if: (i) $h(z)[h(z) - f_0(z)] > 0$; (ii) $[h(z) - f_0(z)][h(x_j) - f_0(x_j)] > 0$; (iii) $h(z)\, h(x_j) > 0$; and (iv) $z \in$ CRITICAL(h) $\cap$ CONSTRAINT(h), then

$$
\begin{aligned}
& h(z) f_0(z) < h(z) f_0(x_j) \\
& \quad \Leftrightarrow 0 < -h(z) f_0(z) + h(z) f_0(x_j) \\
& \quad \Leftrightarrow h(z)[h(z) - h(x_j)] < h(z)[h(z) - f_0(z)] - h(z)[h(x_j) - f_0(x_j)] \qquad (2) \\
& \quad \Leftrightarrow | h(z) - h(x_j)| < |[h(z) - f_0(z)]| - |[h(x_j) - f_0(x_j)]| \\
& \quad \Leftrightarrow \| h \| - |h(x_j)| < \| h - f_0 \| - | h(x_j) - f_0(x_j)|.
\end{aligned}
$$

Then, intuitively, what is taking place in Theorem 4 is that the addition of the correction factor $f(x_j) = h_j(x_j) - h(x_j)$ brings the magnitude of $h(x_j)$ up to the norm value 1 of h, but does not bring the magnitude of $h(x_j) - f_0(x_j)$ up to $\| h - f_0 \|$. As a consequence, x_j is a point of $[a, b]$ increasable in h with respect to f_0.

Before proceeding to a characterization of locally best approximations, we pause to state a lemma that will enable us to take the limit of a sequence of functions constructed by interpolation.

LEMMA 5. *If $\{f_1, ..., f_n\}$ satisfies the Haar condition on $[a, b]$ and if we set $A = \{(x_1, ..., x_n, y_1, ..., y_n) \in E^{2n}: a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b\}$, then the interpolation function $F: A \to$ SPAN($\{f_1, ..., f_n\}$) defined for $(x_1, ..., y_n) \in A$ as the unique function f satisfying $f(x_i) = y_i$, $i = 1, ..., n$, is continuous.*

We are now ready to state and prove our Main Theorem. Notice that not only are additional smoothness assumptions placed on $f_1, ..., f_n$, the basis functions for L, but also on f_0, the function being approximated.

THEOREM 5. *Suppose that:*

(1) *The functions $f_1, ..., f_n$ form a Haar set of continuously differentiable functions on $[a, b]$ (allowing one-sided derivatives at the end points).*

(2) *At each $x \in [a, b]$ there exists an $f \in L =$ SPAN($\{f_1, ..., f_n\}$) satisfying $f(x) = 0$ and $f'(x) \neq 0$.*

(3) *The function $f_0 \notin L$ is a continuously differentiable function on $[a, b]$ whose derivative has zeros comprising a set with finitely many components. (A component of a set is a maximal connected subset.)*

(4) *The best approximation g to f_0 from L satisfies $\| g \| < 1$. Then $h \in S$, the unit sphere of L, is a locally best approximation to f_0 from S if and only if* CONSTRAINT(h) $\cap$ INCR(h, f_0) $= \varnothing$ *and no* $z \in$ CONSTRAINT(h) *with $h'(z) = 0$ is the limit of an infinite sequence $\{x_j\}$ of points in* INCR(h, f_0) *satisfying*

$$h(z) f_0(z) < h(z) f_0(x_j). \qquad (3)$$

Proof. First we prove the "if" part by a contradiction argument. Suppose that h is not a locally best approximation to f_0 from S. If CONSTRAINT(h) $\cap$ INCR(h, f_0) $\neq \varnothing$, then we are through. Otherwise, we may obtain a z and a sequence $\{x_j\}$ from Theorem 4.

It remains only to show that $h'(z) = 0$. To do this, we shall assume that the sequence $\{x_j\}$ has been constructed as per the proof of Theorem 4. By shifting to a subsequence if necessary, no generality is lost in assuming $x_j \notin \{a, b\}$ for all j. (Any point appearing infinitely often would have to be the limit point z.) Thus, x_j is an interior extremum of the better approximation h_j to f_0 from S (better than h), so $h_j'(x_j) = 0$. But differentiation on L, being a linear operator on a finite-dimensional normed linear space, is a continuous operator on L. Hence,

$$\lim_{\substack{f \to 0 \\ f \in L}} \| f' \| = 0.$$

Therefore,

$$0 = \lim_{j \to \infty} (h - h_j)'(x_j) = \lim_{j \to \infty} (h'(x_j) - h_j'(x_j))$$

$$= \lim_{j \to \infty} h'(x_j) = h'(z).$$

Now we proceed to the proof of the converse. Certainly, if CONSTRAINT(h) $\cap$ INCR(h, f_0) $\neq \varnothing$, then h is not a (locally) best approximation to f_0 from S; thus, suppose CONSTRAINT(h) $\cap$ INCR(h, f_0) $= \varnothing$, and suppose $z \in$ CONSTRAINT(h) with $h'(z) = 0$ is the limit of a sequence $\{x_j\}$ in INCR(h, f_0) satisfying (3). We will show that h is not a locally best approximation to f_0 from S.

As a very rough (and inaccurate) plan of attack, we will use the correction functions c_j associated with the x_j to obtain a sequence of better approximations h_j converging to a no worse approximation h_t with $h_t(z) = 1$ and $h_t'(z) \neq 0$ and conclude $\| h_t \| > 1$. (This would indicate that we could finish with Lemma 1.)

Now $h(z) \neq 0$ (since $z \in$ CONSTRAINT(h)), so Lemma 4 yields $z \in$ CRITICAL(h) with $h(z)[h(z) - f_0(z)] > 0$. Then by continuity, there is an open interval I_1 (relative to $[a, b]$) containing z and satisfying both

$$h(z)\, h(x) > 0, \qquad \text{for all } x \in \bar{I}_1 \text{ (the closure of } I_1\text{)}, \tag{4}$$

and

$$h(z)[h(x) - f_0(x)] > 0, \qquad \text{for all } x \in \bar{I}_1\,; \tag{5}$$

that is to say, both h and $h - f_0$ have the same sign throughout $\bar{I}_1$.

Furthermore, if we consider $h(z) f_0(x)$ as a function of x, then for sufficiently small $\epsilon > 0$ there must be an interval $[z - \epsilon, z)$ or $(z, z + \epsilon]$ containing a subsequence of $\{x_j\}$ and in which x satisfies

$$h(z) f_0(z) < h(z) f_0(x). \tag{6}$$

Were this not true, the points where $h(z) f_0(z) = h(z) f_0(x)$ would interlace infinitely often the points x_j where inequality (3) holds. This would give rise to infinitely many local extrema in the function $h(z) f_0(x)$, which in turn would lead to infinitely many components of $(f_0')^{-1}(\{0\})$, in violation of Hypothesis 3.

We will denote by I the intersection of such an interval with the interval $\bar{I}_1$ so that inequalities (4)–(6) hold for all x in the half-open interval I.

Then, using these three inequalities, we may repeat the argument of (2), replacing x_j by x to obtain

$$| h(z) - h(x)| < | h(z) - f_0(z)| - | h(x) - f_0(x)|, \qquad \text{for all } x \in I. \tag{7}$$

Therefore, $I \cap \text{CRITICAL}(h) = \varnothing$, because inequality (7) implies that

$$0 < \| h - f_0 \| - | h(x) - f_0(x)|, \qquad \text{for all } x \in I.$$

By shifting to a subsequence of $\{x_j\}$ and relabeling, we have that x_j is an interior point of I for all j.

Let c_1 denote a correction function of h increasing h at x_1; then $\text{sgn}(h(x_1)) = \text{sgn}(c_1(x_1)) \neq 0$. Let $z_1, ..., z_m$ denote the simple zeros of c_1 not occurring between x_1 and z inclusively. (Note that an odd number of simple zeros must occur between x_1 and z.) Using $[\cdot]$ to denote the greatest integer function, we set $p = 2[(n - 2 - m)/2]$ and select p distinct points $z_{m+1}, ..., z_{m+p}$ in I, distinct from $z_1, ..., z_m$ and not between x_1 and z inclusively. Either $m + p + 1 = n - 1$ or $n - 2$.

We are now in a position to construct by interpolation a one parameter family of correction functions c_x.

Case I. $(m + p + 1 = n - 1)$. For each x between x_1 and z inclusively, let c_x be the unique function in L interpolating zero at $x, z_1, ..., z_{m+p}$ and interpolating $c_1(x_1)$ at x_1.

Case II. $(m + p + 1 = n - 2;\ z \notin \{a, b\})$. For each x between x_1 and z inclusively, let d_x be the unique function interpolating zero at $x, z_1, ..., z_{m+p}$, interpolating 1 at a, and interpolating $(-1)^{n-2}$ at b. Then $c_x(x) = c_1(x_1)\, d_x(x)\, d_x(x_1)$ satisfies $\text{sgn}(c_x(x_1)) = \text{sgn}(c_1(x_1))$.

Case III. $(m + p + 1 = n - 2;\ z \in \{a, b\})$. For each x between x_1 and z inclusively, let c_x be the unique function in L interpolating zero at

$z_1, \ldots, z_{m+p}$, interpolating $|x - z| \cdot c_1(z)$ at z, interpolating $c_1(x_1)$ at x_1 (so c_x has an additional simple zero between z and x_1), and interpolating $(-1)^{n-2} c_1(z)$ at a (if $z \neq a$; at b if $z = a$).

Note in each case that $c_z(z) = 0$, and for each x between z and x_1, exclusively, c_x is a correction function because c_x and c_1 agree in sign on the set $[a, b] - I$, where all the critical points of h lie. Now we set about to obtain "better approximations" h_x to correspond to the correction functions c_x.

First, let f denote a function satisfying $f(z) = 0$ and $f'(z) \neq 0$. (Such a function exists by Hypothesis 2.) Second, select an open interval (relative to $[a, b]$), call it I_2, of length less than $|x_1 - z|$, contained in I_1 and such that z is the only zero of f in I_2. Let I_3 denote the closed interval with end points z and x_1. Then $F: I_3 \to L$ defined by $F(x) = c_x$ is continuous by Lemma 5. Thus, the function $E: I_3 \to E^1$ (Euclidean space) defined by $E(x) = \sup\{\epsilon > 0: \| h - f_0 \| > \sup\{| h(y) + \epsilon c_x(y) - f_0(y)| : y \in [a, b] - I_2\}\}$ is continuous on I_3. Furthermore, the function E is nonzero, since every c_x is a correction function on $[a, b] - I_2$. Then $e_1 = \inf\{E(x): x \in I_3\} > 0$ since the infimum is attained by continuity on the compact interval I_3.

Choose e with $0 < e < e_1$ so that $\| h + ec_{x_1} - f_0 \| < \| h - f_0 \|$. Further, we may assume that $\| h + ec_{x_1} \| < 1$, for were this not true for all e sufficiently small, then (since c_{x_1} is a correction function, decreasing the error function) h would not be a locally best approximation to f_0 (which is what we are trying to show).

By choosing a small (possibly negative) multiple of f, if necessary, and relabeling, we may further assume that f satisfies in addition to $f(z) = 0$ and $f'(z) \neq 0$:

(i) $\| f \| < 1 - \| h + ec_{x_1} \|$,

(ii) $f'(z)[c_1(x_1) - c_1(z)]/(x_1 - z) > 0$,

(iii) $ec_z'(z) + f'(z) \neq 0$.

Define h_x for $x \in I_3$ by $h_x = h + ec_x + f$, then $H: I_3 \to L$ defined by $H(x) = h_x$ is continuous. Furthermore, recalling that $c_z(z) = 0$, we have that $|h_z(z)| = 1$; and by allowing one-sided derivatives at end points we see that

$$h_z'(z) = h'(z) + ec_z'(z) + f'(z) = ec_z'(z) + f'(z) \neq 0.$$

In fact,

$$\begin{aligned} &h_z'(z)[c_1(x_1) - c_1(z)]/(x_1 - z) \\ &\quad = [ec_z'(z) + f'(z)][c_1(x_1) - c_1(z)]/(x_1 - z) > 0. \end{aligned}$$

This can be seen from (ii) above plus the fact that c_z changes sign from $\operatorname{sgn}(c_1(x_1))$ to zero at z, so that $c_z'(z)[c_1(x_1) - c_1(z)]/(x_1 - z) \geqslant 0$.

Thus, the magnitude of $h_z(y)$ is increasing at $y = z$ as one moves in the direction of x_1. Consequently, $\| h_z \| > 1$.

Since $\| h_{x_1} \| < 1$ (see (i) above), select $x \in I_3$ with $\| h_x \| = 1$; then h_x is a better approximation to f_0 from S than is h.

To prove that h is not a locally best approximation to f_0 from S, we need only note that $h_x - h = ec_x + f$. Since we can choose e and $\|f\|$ as small as we wish, we obtain better approximations arbitrarily close to h. This completes the proof of the theorem.

We conclude with examples intended to illustrate the various facets of the characterization of locally best approximations from S.

EXAMPLE 1. $L = \text{SPAN}(\{f_1, f_2\}) = \text{SPAN}(\{1, x\})$; $[a, b] = [-1, 1]$; $f_0(x) = x^2$. This example is straightforward. Note that L is a two-dimensional Haar space. Furthermore, $g = 1/2$ is the best approximation to f_0 from L. Two locally best approximations to x^2 from S are $h_1(x) = (6 - 4(2)^{1/2})\, x + 4(2)^{1/2} - 5$ and $h_2(x) = h_1(-x) = (4(2)^{1/2} - 6)\, x + 4(2)^{1/2} - 5$. To verify this we argue as follows. We see that

$$\begin{aligned}\| h_1 - f_0 \| &= 12 - 8(2)^{1/2} = -(h_1(-1) - f_0(-1)) \\ &= h_1(3 - 2(2)^{1/2}) - f_0(3 - 2(2)^{1/2}).\end{aligned}$$

Hence, $\text{CRITICAL}(h_1) = \{-1, 3 - 2(2)^{1/2}\}$. By Lemma 2, any correction function must be positive at -1 and negative at $3 - 2(2)^{1/2}$. Hence, its single zero must occur between these points and it must be negative on $[3 - 2(2)^{1/2}, 1] \equiv I$. Since h_1 is positive on I, $I \cap \text{INCR}(h_1, x^2) = \varnothing$. But $\text{CONSTRAINT}(h_1) = \{1\} \subset I$, so that Theorem 5 yields the desired result. The function h_2 is a locally best approximation by symmetry.

Further analysis reveals that $\{h_1, h_2\}$ is the set of best approximations to x^2 from S. We need only note that for all $h \in S - \{f_1, -f_1\}$, either $\text{CONSTRAINT}(h) = \{1\}$, or $\text{CONSTRAINT}(h) = \{-1\}$, so that (by symmetry) if there are other best (or better) approximations, then one must pass through (1, 1). But altering the slope of h_1 will only increase the norm of the error. Note also that altering the slope of h_1 would make CRITICAL(h) a singleton. This would enable correction functions to have a zero between the critical point and 1, and place 1 in $\text{INCR}(h, f_0)$. Theorem 5 would then reveal that such h are not best approximations.

EXAMPLE 2. $L = \text{SPAN}\{f_1, f_2, f_3\} = \text{SPAN}\{1, x, x^2\}$; $[a, b] = [-1, 1]$; $f_0 = \frac{1}{4}x^4$. L is a three-dimensional Haar space. Furthermore, $\| g \| < 1$ since

$$\| g \| \leqslant \| g - f_0 \| + \| f_0 \| \leqslant \| 0 - f_0 \| + \| f_0 \| = 2 \| f_0 \| = \tfrac{1}{2}.$$

Let $h(x) = 1 - x^2$; then CONSTRAINT$(h) = \{0\}$, CRITICAL$(h) = \{0\}$, and INCR$(h, f_0) = [-1, 0) \cup (0, 1]$ (since by Lemma 2 we may construct correction functions with zeros arbitrarily close to $x = 0$). Letting $z = 0$, we have $h(z) f_0(z) < h(z) f_0(x)$ for all $x \neq 0$, so that $x_j = 1/j$, $j = 1, 2,\ldots,$ is a sequence of points in INCR(h, f_0) converging to z and satisfying (3). Thus, by Theorem 5, h is not a best approximation to $\frac{1}{4}x^4$ from S. Indeed, all functions $h_a(x) = -(x - a)^2 + 1$ for a positive and sufficiently small are better approximations.

Finally, consider $k(x) = (7/4)\,x^2 - (3/4)$. Then, CONSTRAINT$(k) = \{-1, 1\}$, CRITICAL$(k) = \{-1, 0, 1\}$, and INCR$(k, f_0) = (-1, 0) \cup (0, 1)$. But $k(z) f_0(z) < k(z) f_0(x)$ becomes $1 \cdot \frac{1}{4} < 1 \cdot \frac{1}{4}x^4$ when $z = \pm 1$. Since this is not true for $x \in (-1, 1)$ there is no sequence $\{x_j\}$ to satisfy the hypotheses of Theorem 5. Hence, k is a locally best approximation to $\frac{1}{4}x^4$ from S.

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